

New proofs for the two Barnes lemmas and an additional lemma

Bernd Jantzen*

*Institut für Theoretische Teilchenphysik und Kosmologie,
RWTH Aachen University, 52056 Aachen, Germany*

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Mellin–Barnes (MB) representations have become a widely used tool for the evaluation of Feynman loop integrals appearing in perturbative calculations of quantum field theory. Some of the MB integrals may be solved analytically in closed form with the help of the two Barnes lemmas which have been known in mathematics already for one century. The original proofs of these lemmas solve the integrals by taking infinite series of residues and summing these up via hypergeometric functions. This paper presents new, elegant proofs for the Barnes lemmas which only rely on the well-known basic identity of MB representations, avoiding any series summations. They are particularly useful for presenting and proving the Barnes lemmas to students of quantum field theory without requiring knowledge on hypergeometric functions. The paper also introduces and proves an additional lemma for a MB integral $\int dz$ involving a phase factor $\exp(\pm i\pi z)$.

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I. INTRODUCTION

When working with generalized hypergeometric functions and relating them through complex contour integrals, about one century ago, Barnes introduced the two following identities: The *first Barnes lemma* [1],

$$\begin{aligned} & \int_{-i\infty}^{i\infty} \frac{dz}{2i\pi} \Gamma(\alpha_1 - z) \Gamma(\alpha_2 - z) \Gamma(\beta_1 + z) \Gamma(\beta_2 + z) \\ &= \frac{\Gamma(\alpha_1 + \beta_1) \Gamma(\alpha_1 + \beta_2) \Gamma(\alpha_2 + \beta_1) \Gamma(\alpha_2 + \beta_2)}{\Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)}, \end{aligned} \quad (1)$$

and the *second Barnes lemma* [2],

$$\begin{aligned} & \int_{-i\infty}^{i\infty} \frac{dz}{2i\pi} \frac{\Gamma(\alpha_1 - z) \Gamma(\alpha_2 - z) \Gamma(\beta_1 + z) \Gamma(\beta_2 + z)}{\Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \beta_3 + z)} \\ & \quad \times \Gamma(\beta_3 + z) \\ &= \frac{\Gamma(\alpha_1 + \beta_1) \Gamma(\alpha_1 + \beta_2) \Gamma(\alpha_1 + \beta_3)}{\Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_3)} \\ & \quad \times \frac{\Gamma(\alpha_2 + \beta_1) \Gamma(\alpha_2 + \beta_2) \Gamma(\alpha_2 + \beta_3)}{\Gamma(\alpha_1 + \alpha_2 + \beta_2 + \beta_3)}. \end{aligned} \quad (2)$$

These identities involve so-called Barnes contour integrals whose paths of integration run parallel to the imaginary axis and are curved where necessary to ensure that they separate positive and negative sequences of poles in the integrands. Explicitly, the *right poles* of gamma functions of the type $\Gamma(\alpha_i - z)$ (at $z = \alpha_i, \alpha_i + 1, \dots$) must lie to the right of the integration contour, whereas the *left poles* of gamma functions $\Gamma(\beta_j + z)$ (at $z = -\beta_j, -\beta_j - 1, \dots$) lie to the left of it. The two Barnes lemmas are valid whenever such an integration contour

can be found, i.e. unless any of the $\alpha_i + \beta_j$ is zero or a negative integer. Due to the asymptotic behavior of the gamma function (see, e.g., Eq. 8.328 1. in [3]),

$$|\Gamma(x + iy)| \underset{|y| \rightarrow \infty}{\sim} \sqrt{2\pi} e^{-|y|\pi/2} |y|^{x-1/2} \quad [x, y \in \mathbb{R}], \quad (3)$$

the integrands in (1) and (2) are exponentially damped for $z \rightarrow \pm i\infty$ and the integrals converge.

In the context of loop calculations in quantum field theory, Barnes contour integrals arise when using Mellin–Barnes (MB) representations by applying the basic identity

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_{-i\infty}^{i\infty} \frac{dz}{2i\pi} \frac{B^z}{A^{\lambda+z}} \Gamma(-z) \Gamma(\lambda + z), \quad (4)$$

with the above-mentioned definition of the integration contour. In particular, if $\text{Re } \lambda > 0$, the contour can be chosen as a straight line in the strip $-\text{Re } \lambda < \text{Re } z < 0$. When the MB integration in (4) is interchanged with other integrations over parameters contained in A or B , the convergence of these integrations will usually be encoded in the arguments of the resulting gamma functions. The change in the order of integrations is justified if the real parts of the arguments of all gamma functions remain positive along the path of the MB integration. Such MB integrals may be analytically continued to parameter values $\text{Re } \lambda < 0$, provided that the MB integration contours are deformed in accordance with the requirements for Barnes contour integrals described above, separating right and left poles from all gamma functions in the integrand.

The identity (4) is valid unless λ is a nonpositive integer. Also the ratio B/A must be away from the negative real axis, otherwise the exponential damping of the gamma functions for $z \rightarrow i\infty$ or $z \rightarrow -i\infty$ is compensated by $(B/A)^z \propto e^{\pm i\pi z}$ and the integral (4) might diverge at one of its boundaries.

* jantzen@physik.rwth-aachen.de

For $|A| > |B|$, the MB integration contour can be closed to the right at $\text{Re } z = +\infty$, and the series of residues at $z = 0, 1, \dots$ reproduces the Taylor expansion of the left-hand side of (4) for $|A| > |B|$. On the other hand, for $|A| < |B|$, the contour may be closed to the left at $\text{Re } z = -\infty$, and the series of residues at $z = -\lambda, -\lambda - 1, \dots$ coincides with the expansion for $|A| < |B|$. Therefore the MB representation (4) provides a way of separating a sum of two terms A and B without requiring a particular hierarchy between them.

The MB representation (4) was applied originally [4] in order to turn massive Feynman propagators $1/(k^2 - m^2 + i0)$ into massless ones by separating the mass term from the squared momentum, using (4) with $A = k^2$ and $B = -m^2 + i0$ (see also, e.g., [5–7] as early references). Later on parametric integrals derived from Feynman diagrams have been evaluated by applying as many MB representations as necessary to obtain results in terms of gamma functions under the MB integrals (see, e.g., [8–11]). The construction of such MB representations for Feynman integrals with planar topologies is automated by the Mathematica tool **AMBRE** [12, 13].

When evaluating the MB integrations, one has to take care of singularities occurring in certain parameter limits (like $d \rightarrow 4$ in dimensional regularization). These singularities arise when a right pole comes arbitrarily close to a left pole, pinching the integration contour which separates them. Two strategies [14, 15] were formulated to systematically extract such singularities analytically. By taking a finite number of residues, they both end up with integration contours which are not pinched any more between right and left poles. These contours can then be chosen as straight lines parallel to the imaginary axis and the MB integrations may be performed numerically. The strategy of [15] is automated in the private computer code [16] and by the Mathematica package **MB** [17]. More recently also the strategy of [14] has been implemented in the computer program **MBresolve** [18].

MB representations are also employed for the asymptotic expansion of Feynman integrals (see, e.g., [8–11, 19–22]). If the MB integral $\int dz$ involves the factor t^z , its asymptotic expansion in the limit $t \rightarrow 0$ is obtained by picking up, order by order, the residues on the right-hand side of the integration contour. Similarly the asymptotic expansion in the limit $t \rightarrow \infty$ results from the residues on the left-hand side of the contour. Czakon’s code **MBasymptotics** [23] performs this task automatically. The contributions obtained from asymptotically expanding MB representations often correspond to those arising in the strategy of expansion by regions [24–26]. This correspondence is emphasized and used for cross-checks in [21, 22]. Also combinations of expansion by regions with further MB representations for evaluating the contributions have proven fruitful [27].

The application of MB representations is explained in many details in the book [28]. Computer codes are summarized on the web site [23].

When dealing with multifold MB representations, it

is always preferable to eliminate as many MB integrals as possible using the Barnes lemmas (1) and (2) before proceeding to the extraction of singularities and numerical integrations. So these two lemmas have become widely used in the application of MB representations. They are implemented, in addition to previously mentioned tools, in Kosower’s Mathematica package **Barnes Routines** [23]. Let us now turn to the proofs of the two Barnes lemmas.

In his original papers, Barnes focuses on the treatment of hypergeometric functions. So, naturally, his proofs of the two lemmas employ properties of such functions. For proving the first lemma (1), Barnes [1] closes the integration contour to one side, which is only possible for $\text{Re}(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) < 1$, and sums the two infinite series of residues from the corresponding poles. These series are identified with hypergeometric ${}_2F_1$ functions of unit argument. After relating the ${}_2F_1$ functions to gamma functions, using several times $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$, and employing a trigonometric identity, Barnes arrives at the right-hand side of (1). The validity of the lemma for general complex values of $\alpha_{1,2}$ and $\beta_{1,2}$ is argued by analytic continuation.

In his proof of the second lemma (2), Barnes [2] starts with the series representation of a hypergeometric ${}_3F_2$ function of unit argument. The number of gamma functions depending on the summation index is reduced by two using the first Barnes lemma (1) from right to left. The series is then identified with a ${}_2F_1$ function of unit argument which is written in terms of gamma functions. The resulting expression involves a contour integral which can be brought into agreement with the left-hand side of (2). This relates the integral from the second lemma to the ${}_3F_2$ function, which, for the specific choice of parameters present in (2), however, reduces to a ${}_2F_1$ function of unit argument and can therefore be expressed through gamma functions, producing the right-hand side of the second lemma.

These two original proofs by Barnes have been reproduced in several books, e.g., [29–31]. Their understanding requires knowledge on the series representations and other properties of hypergeometric functions which appear in intermediary steps. In the current paper, however, I present new, elegant proofs for the two Barnes lemmas which only use integral transformations in a straightforward way, avoiding completely the series summations which the original proofs employ. My new proofs merely rely on the basic identity of MB representations (4) and on the integral representations of the gamma function and Euler’s beta function. The latter is well known in the field of loop calculations. Therefore, when introducing MB representations and the Barnes lemmas to students of quantum field theory, I suggest to use the proofs presented here rather than the original ones by Barnes.

The new proofs for the two Barnes lemmas are found in Secs. II and III, respectively. Sec. IV introduces and proves an additional lemma for a MB integral involving

a phase factor $e^{\pm i\pi z}$, which, apart from its application to loop calculations, is of particular interest for studying the convergence behavior of such integrals at their boundaries $\pm i\infty$. Finally conclusions are presented in Sec. V.

II. PROOF OF THE FIRST BARNES LEMMA

We start with the left-hand side of the first Barnes lemma (1),

$$\int_{-i\infty}^{i\infty} \frac{dz}{2i\pi} \Gamma(\alpha_1 - z) \Gamma(\alpha_2 - z) \Gamma(\beta_1 + z) \Gamma(\beta_2 + z). \quad (5)$$

Let us assume that

$$\operatorname{Re}(\alpha_i + \beta_j) > 0 \forall i, j \quad (6)$$

such that the integration contour can be chosen as a straight line parallel to the imaginary axis with

$$-\operatorname{Re} \beta_j < \operatorname{Re} z < \operatorname{Re} \alpha_i \forall i, j. \quad (7)$$

We insert for $\Gamma(\alpha_1 - z)$ and $\Gamma(\beta_1 + z)$ the corresponding integral representations of the gamma function (see, e.g., Eq. 8.310 1. in [3]),

$$\Gamma(\alpha) = \int_0^\infty dt t^{\alpha-1} e^{-t} \quad [\operatorname{Re} \alpha > 0], \quad (8)$$

and obtain

$$\begin{aligned} & \int_0^\infty dt_1 t_1^{\alpha_1-1} e^{-t_1} \int_0^\infty dt_2 t_2^{\beta_1-1} e^{-t_2} \\ & \times \int_{-i\infty}^{i\infty} \frac{dz}{2i\pi} \left(\frac{t_2}{t_1}\right)^z \Gamma(\alpha_2 - z) \Gamma(\beta_2 + z) \end{aligned} \quad (9)$$

from (5). The change in the order of the integrations is justified by the condition (7) for the contour. The basic MB identity (4) can easily be reformulated by a shift in the integration variable:

$$\int_{-i\infty}^{i\infty} \frac{dz}{2i\pi} \left(\frac{B}{A}\right)^z \Gamma(\alpha - z) \Gamma(\beta + z) = \frac{\Gamma(\alpha + \beta) B^\alpha A^\beta}{(A + B)^{\alpha + \beta}}. \quad (10)$$

Inserting (10) for the contour integral in (9) yields

$$\Gamma(\alpha_2 + \beta_2) \int_0^\infty dt_1 \int_0^\infty dt_2 \frac{t_1^{\alpha_1 + \beta_2 - 1} t_2^{\alpha_2 + \beta_1 - 1}}{(t_1 + t_2)^{\alpha_2 + \beta_2}} e^{-(t_1 + t_2)}. \quad (11)$$

This double integral factorizes by performing the variable transformation $t_1 = \eta \xi$ and $t_2 = \eta(1 - \xi)$:

$$\begin{aligned} & \Gamma(\alpha_2 + \beta_2) \int_0^\infty d\eta \eta^{\alpha_1 + \beta_1 - 1} e^{-\eta} \\ & \times \int_0^1 d\xi \xi^{\alpha_1 + \beta_2 - 1} (1 - \xi)^{\alpha_2 + \beta_1 - 1}. \end{aligned} \quad (12)$$

While the η -integral is given by (8), the ξ -integral is a representation of Euler's beta function (see, e.g., Eqs. 8.380 1. and 8.384 1. in [3]),

$$\int_0^1 d\xi \xi^{\alpha_1 - 1} (1 - \xi)^{\alpha_2 - 1} = \frac{\Gamma(\alpha_1) \Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)} \quad [\operatorname{Re} \alpha_i > 0], \quad (13)$$

which is an identity often used in loop calculations. This leads to the result

$$\frac{\Gamma(\alpha_1 + \beta_1) \Gamma(\alpha_1 + \beta_2) \Gamma(\alpha_2 + \beta_1) \Gamma(\alpha_2 + \beta_2)}{\Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)}, \quad (14)$$

in agreement with the right-hand side of (1), and thus proves the first Barnes lemma.

The transformations in this proof are valid if the condition (6) is fulfilled, which is when the integration contour can be chosen as a straight line (7) or, equivalently, when the real parts of the arguments of all gamma functions in the numerator of the result (14) are positive. Through analytic continuation, the validity of the first Barnes lemma may be extended to general complex values of $\alpha_{1,2}$ and $\beta_{1,2}$, with the exception of parameter choices where no integration contour separating right and left poles is found in (5), which corresponds to the right-hand side (14) being ill-defined as well when at least one of the $\alpha_i + \beta_j$ is a nonpositive integer.

III. PROOF OF THE SECOND BARNES LEMMA

The left-hand side of the second Barnes lemma (2) reads

$$\begin{aligned} & \int_{-i\infty}^{i\infty} \frac{dz}{2i\pi} \frac{\Gamma(\alpha_1 - z) \Gamma(\alpha_2 - z) \Gamma(\beta_1 + z) \Gamma(\beta_2 + z)}{\Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \beta_3 + z)} \\ & \times \Gamma(\beta_3 + z). \end{aligned} \quad (15)$$

As for the other lemma, we assume that the condition (6) holds for all pairs of parameters such that the integration contour can be chosen as a straight line in the strip defined by (7). Let us employ the first Barnes lemma (1) to replace three of the gamma functions in the integrand:

$$\begin{aligned} & \frac{\Gamma(\beta_2 + z) \Gamma(\beta_3 + z)}{\Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \beta_3 + z)} \\ & = \frac{1}{\Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_3)} \\ & \times \int_{-i\infty}^{i\infty} \frac{ds}{2i\pi} \Gamma(z - s) \Gamma(\alpha_1 + \alpha_2 + \beta_1 - s) \\ & \times \Gamma(\beta_2 + s) \Gamma(\beta_3 + s). \end{aligned} \quad (16)$$

Also this second contour integral over s can be chosen along a straight line parallel to the imaginary axis with

$$-\operatorname{Re} \beta_j < \operatorname{Re} s < \operatorname{Re} z < \operatorname{Re} \alpha_i \forall i, j. \quad (17)$$

Then the order of the integrations may safely be interchanged and the z -integral yields

$$\int_{-i\infty}^{i\infty} \frac{dz}{2i\pi} \Gamma(\alpha_1 - z) \Gamma(\alpha_2 - z) \Gamma(\beta_1 + z) \Gamma(-s + z) = \frac{\Gamma(\alpha_1 + \beta_1) \Gamma(\alpha_2 + \beta_1) \Gamma(\alpha_1 - s) \Gamma(\alpha_2 - s)}{\Gamma(\alpha_1 + \alpha_2 + \beta_1 - s)}, \quad (18)$$

using the first Barnes lemma (1) again. Also the final s -integral is solved via the first lemma,

$$\int_{-i\infty}^{i\infty} \frac{ds}{2i\pi} \Gamma(\alpha_1 - s) \Gamma(\alpha_2 - s) \Gamma(\beta_2 + s) \Gamma(\beta_3 + s) = \frac{\Gamma(\alpha_1 + \beta_2) \Gamma(\alpha_1 + \beta_3) \Gamma(\alpha_2 + \beta_2) \Gamma(\alpha_2 + \beta_3)}{\Gamma(\alpha_1 + \alpha_2 + \beta_2 + \beta_3)}. \quad (19)$$

We obtain the result

$$\frac{\Gamma(\alpha_1 + \beta_1) \Gamma(\alpha_1 + \beta_2) \Gamma(\alpha_1 + \beta_3)}{\Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) \Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_3)} \times \frac{\Gamma(\alpha_2 + \beta_1) \Gamma(\alpha_2 + \beta_2) \Gamma(\alpha_2 + \beta_3)}{\Gamma(\alpha_1 + \alpha_2 + \beta_2 + \beta_3)}, \quad (20)$$

which coincides with the right-hand side of the second Barnes lemma (2). So we have proven the second Barnes lemma simply by applying three times the first Barnes lemma.

As for the first lemma, the condition (6) can be relaxed to general complex values of the parameters, as long as none of the $\alpha_i + \beta_j$ is a nonpositive integer. And, as before, the arguments of the gamma functions in the numerator of (20) reflect the singularity structure of the integral (15).

IV. ADDITIONAL LEMMA FOR MELLIN–BARNES INTEGRALS INVOLVING A PHASE FACTOR

As discussed in the introduction (Sec. I), the exponential damping of the gamma function towards $\pm i\infty$ (3) usually makes Barnes contour integrals converge well and allows for numerical integrations of such integrals. This nice feature is potentially spoiled by phase factors $e^{\pm i\pi z}$ in the integrand. Take, for instance, the basic MB identity (4) with $A = 1$ and $B = -\rho + i0$ ($\rho > 0$), thus $B^z = \rho^z e^{i\pi z}$. The integrand still vanishes exponentially for $z \rightarrow i\infty$, but at the lower boundary it behaves as

$$|\rho^z e^{i\pi z} \Gamma(-z) \Gamma(\lambda + z)| \underset{z \rightarrow -i\infty}{\sim} 2\pi \rho^{\operatorname{Re} z} |\operatorname{Im} z|^{\operatorname{Re} \lambda - 1}, \quad (21)$$

so the integral only converges for $\operatorname{Re} \lambda < 0$. This divergence is reflected in the left-hand side of (4), here $1/(1 - \rho + i0)^\lambda$, which exhibits a singular cancelation for $\rho \rightarrow 1$ if $\operatorname{Re} \lambda > 0$.

When an “ordinary” MB integral (which converges at $z = \pm i\infty$) is solved in closed form, as for the two Barnes lemmas (1) and (2), its singularities are encoded in the arguments of the resulting gamma functions. These singularities originate from right and left poles pinching the integration contour between them (see Sec. I). It is desirable to establish solutions in closed form also for MB integrals with phase factors $e^{\pm i\pi z}$ where a potential divergence at $z = \pm i\infty$ is parametrized by the resulting gamma functions as well.

One such identity is provided by the following *additional lemma*:

$$\int_{-i\infty}^{i\infty} \frac{dz}{2i\pi} e^{\pm i\pi z} \frac{\Gamma(\alpha - z) \Gamma(\beta_1 + z) \Gamma(\beta_2 + z)}{\Gamma(\gamma + z)} = e^{\pm i\pi \alpha} \frac{\Gamma(\alpha + \beta_1) \Gamma(\alpha + \beta_2) \Gamma(\gamma - \alpha - \beta_1 - \beta_2)}{\Gamma(\gamma - \beta_1) \Gamma(\gamma - \beta_2)}. \quad (22)$$

While the first two gamma functions in the numerator of the right-hand side of (22) reflect the singularities originating from right and left poles, the third gamma function parametrizes the divergence at $z = \mp i\infty$. In the case of the phase factor $e^{+i\pi z}$, the integrand in (22) exhibits the asymptotic behavior

$$\left| e^{i\pi z} \frac{\Gamma(\alpha - z) \Gamma(\beta_1 + z) \Gamma(\beta_2 + z)}{\Gamma(\gamma + z)} \right| \underset{z \rightarrow -i\infty}{\sim} 2\pi |\operatorname{Im} z|^{\operatorname{Re}(\alpha + \beta_1 + \beta_2 - \gamma) - 1}. \quad (23)$$

So the integral only converges at $z = -i\infty$ for

$$\operatorname{Re}(\gamma - \alpha - \beta_1 - \beta_2) > 0, \quad (24)$$

which matches the argument of the corresponding gamma function in the right-hand side of (22).

Let us turn to the proof of the additional lemma (22). In analogy to the previous proofs we assume

$$\operatorname{Re}(\alpha + \beta_j) > 0 \quad \forall j \quad (25)$$

such that the integration contour can be chosen as a straight line parallel to the imaginary axis with

$$-\operatorname{Re} \beta_j < \operatorname{Re} z < \operatorname{Re} \alpha \quad \forall j. \quad (26)$$

Here we also have to require the condition (24) for the convergence of the integral at both boundaries $z = \pm i\infty$. We start with the left-hand side of (22) and replace two gamma functions using (13):

$$\frac{\Gamma(\beta_2 + z)}{\Gamma(\gamma + z)} = \frac{1}{\Gamma(\gamma - \beta_2)} \int_0^1 d\xi \xi^{\beta_2 + z - 1} (1 - \xi)^{\gamma - \beta_2 - 1}. \quad (27)$$

The convergence of the integral (27) follows from (24) and (26). We may safely change the order of integrations

and perform the contour integral first employing the basic MB identity in the form (10):

$$\begin{aligned} & \int_{-i\infty}^{i\infty} \frac{dz}{2i\pi} \underbrace{e^{\pm i\pi z} \xi^z}_{(-\xi \pm i0)^z} \Gamma(\alpha - z) \Gamma(\beta_1 + z) \\ &= \Gamma(\alpha + \beta_1) \underbrace{(-\xi \pm i0)^\alpha}_{e^{\pm i\pi\alpha} \xi^\alpha} (1 - \xi)^{-\alpha - \beta_1}. \end{aligned} \quad (28)$$

Then the ξ -integral is given by (13):

$$\begin{aligned} & \int_0^1 d\xi \xi^{\alpha + \beta_2 - 1} (1 - \xi)^{\gamma - \alpha - \beta_1 - \beta_2 - 1} \\ &= \frac{\Gamma(\alpha + \beta_2) \Gamma(\gamma - \alpha - \beta_1 - \beta_2)}{\Gamma(\gamma - \beta_1)}. \end{aligned} \quad (29)$$

By assembling the pieces from (27), (28), and (29) we obtain the right-hand side of (22).

The validity of the additional lemma (22) can be extended via analytic continuation to general complex values of α , $\beta_{1,2}$, and γ by relaxing the condition (25), as discussed for the two previous lemmas. However, the condition (24) is crucial for the convergence of the contour integral and cannot be waived.

V. CONCLUSIONS

New, elegant proofs have been presented for the two Barnes lemmas which are often used in loop calculations in order to solve MB integrals. The new proofs avoid the series summations and representations of hypergeometric functions which are used in the original proofs by Barnes. They are well suited for courses in quantum field theory treating MB representations because they are only based on identities generally known there.

Also an additional lemma has been introduced and proven which parametrizes the divergence of a MB integral $\int dz$ when a phase factor $e^{\pm i\pi z}$ is present. This new lemma has been used by myself in many evaluations of Feynman integrals, and other authors have probably solved this and similar MB integrals as well, e.g. by taking a series of residues and summing them up via a hypergeometric function. But, to my knowledge, this lemma has not been published in closed form so far.

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